About conflict in the theory of belief functions

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Abstract In the theory of belief functions, the conflict is an important concept. Indeed, combining several imperfect experts or sources allows conflict. However, the mass appearing on the empty set during the conjunctive combination rule is generally considered as conflict, but that is not really a conflict. Some measures of conflict have been proposed, we recall some of them and we show some counter-intuitive examples with these measures. Therefore we define a conflict measure based on expected properties. This conflict measure is build from the distance-based conflict measure weighted by a degree of inclusion introduced in this paper.

1 Introduction

The theory of belief functions was first introduced by [2] in order to represent some imprecise probabilities with *upper* and *lower probabilities*. Then [13] proposed a mathematical theory of evidence with is now widely used for information fusion. Combining imperfect sources of information leads inevitably to conflict. One can consider that the conflict comes from the non-reliability of the sources or the sources do not give information on the same observation. In this last case, one must not combine them.

Let $\Theta = \{\theta_1, \dots, \theta_n\}$ be a frame of discernment of exclusive and exhaustive hypothesis. A mass function m is the mapping from elements of the power set 2^Θ onto $[0,1]$ such that:

$$\sum_{X \in 2^\Theta} m(X) = 1. \tag{1}$$

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A focal element X is an element of 2^Θ such that $m(X) \neq 0$. If the focal elements are nested, the mass functions is *consonant*. Constraining $m(\emptyset) = 0$ corresponds to a closed-world assumption [13], while allowing $m(\emptyset) \geq 0$ corresponds to an open world assumption [15]. Smets interprets this mass on the empty set such as an non-expected hypothesis and normalizes it in the pignistic probability defined for all $X \in 2^\Theta$, with $X \neq \emptyset$ by:

$$\mathrm{BetP}(X) = \sum_{Y \in 2^\Theta, Y \neq \emptyset} \frac{|X \cap Y|}{|Y|} \frac{m(Y)}{1 - m(\emptyset)}. \tag{2}$$

The first combination rule has been proposed by Dempster [2] and is defined for two mass functions m_1 and m_2, for all $X \in 2^\Theta$, with $X \neq \emptyset$ by:

$$m_{\mathrm{DS}}(X) = \frac{1}{1-k} \sum_{A \cap B = X} m_1(A) m_2(B), \tag{3}$$

where $k = \sum_{A \cap B = \emptyset} m_1(A) m_2(B)$ is the inconsistence of the combination and generally called conflict. We call it here the *global conflict*.

To stay in an open world, Smets [15] proposes the non-normalized conjunctive rule given for two mass functions m_1 and m_2 and for all $X \in 2^\Theta$ by:

$$m_{\mathrm{Conj}}(X) = \sum_{A \cap B = X} m_1(A) m_2(B) := (m_1 \bigcirc\!\!\!\!\cap\; m_2)(X). \tag{4}$$

These both rules allow to reduce the imprecision of the focal elements and to increase the belief on concordant elements. The main assumptions to apply these rules are the cognitive independence and the reliability of the sources.

Based on the results of these rules, the problem enlightened by the famous Zadeh's example [20] is the repartition of the global conflict. Indeed, consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and two experts opinions given by $m_1(\theta_1) = 0.9$, $m_1(\theta_3) = 0.1$, and $m_2(\theta_2) = 0.9$, $m_1(\theta_3) = 0.1$, the mass function resulting in the combination using Dempster's rule is $m(\theta_3) = 1$ and using conjunctive rule is $m(\emptyset) = 0.99$, $m(\theta_3) = 0.01$. Therefore, several combination rules have been proposed to manage this global conflict [16, 9].

As observed in [8, 10], the weight of conflict given by $k = m_{\mathrm{Conj}}(\emptyset)$ is not a conflict measure between the mass functions. Indeed, the conjunctive-based rules are not idempotent (as the majority of the rules defined to manage the global conflict): the combination of identical mass functions leads generally to a positive value of k. Hence, new kind of conflict measures are defined in [10].

In the following section 2, we recall the different measures of conflict in the theory of belief functions. Then, on the bases of wanted properties we propose a new conflict measure based on a degree of inclusion that we define in section 3. The last section 4 presents the interest of the proposed conflict measures on numerical example and gives uses of this measure.

2 Conflict measures

First of all, we should not mix up conflict measure and contradiction measure. The measures defined by [7, 17] are not conflict measures, but some discord and specificity measures (to take the terms of [6]) we call contradiction measures. We define the contradiction and conflict measures by the following definitions:

Definition *A* contradiction *in the theory of belief functions quantifies how a mass function contradicts itself.*

Definition (C1) *The* conflict *in the theory of belief functions can be defined by the contradiction between two or more mass functions.*

Therefore, is the mass on the empty set or the functions of this mass (such as $-\ln(1-m_{\text{Conj}}(\emptyset))$ proposed by [13]) a conflict measure? It seems obvious that the property of the non-idempotence is a problem to use this as a conflict measure. However, if we define a conflict measure such as $\text{Conf}(m_1, m_2) = m_{\text{Conj}}(\emptyset)$, we note that $\text{Conf}(m_1, m_\Theta) = 0$ where $m_\Theta(\Theta) = 1$ is the ignorance. Indeed, the ignorance is the neutral element for the conjunctive combination rule. This propertiy seems to be reached by a conflict measure.

Other conflict measures have been defined. In [5], a conflict measure is given by:

$$\text{Conf}(m_1, m_2) = 1 - \frac{\mathbf{pl}_1^T.\mathbf{pl}_2}{\|\mathbf{pl}_1\|\|\mathbf{pl}_2\|} \tag{5}$$

where $\mathbf{pl}$ is the plausivity function and $\mathbf{pl}_1^T.\mathbf{pl}_2$ the vector product in 2^n space of both plausibility functions. However, generally $\text{Conf}(m_1, m_\Theta) \neq 0$, that seems couter-intuitive.

Auto-conflict

Introduced by [11], the auto-conflict of order s for one expert is given by:

$$a_s = \left(\mathop{\bigcirc\!\!\!\!\!\cap}_{i=1}^{s} m \right)(\emptyset). \tag{6}$$

where $\bigcirc\!\!\!\!\!\cap$ is the conjunctive operator of Equation (4). The following property holds: $a_s \leq a_{s+1}$, meaning that due to the non-indempotence of $\bigcirc\!\!\!\!\!\cap$, the more m is combined with itself the nearer to 1 k is, and so in a general case, the more the number of experts is high the nearer to 1 k is. The behavior of the auto-conflict was studied in [10] and show that we should take into account the auto-conflict in the global conflict in order to really define a conflict. In [19], the auto-conflict was defined and called the plausibility of the belief structure with itself. The auto-conflict is a kind of measure of the contradiction, but depends on the order. A measure of contradiction independent on the order has been defined in [14].

Conflict measure based on a distance

The definition of the conflict (C1) involves firstly to measure it on the bba's space and secondly that if the opinions of two experts are far from each other, we consider that they are in conflict. That suggests a notion of distance. That is the reason why in [10], we give a definition of the measure of conflict between experts assertions through a distance between their respective bba's. The conflict measure between 2 experts is defined by:

$$\mathrm{Conf}(1,2) = d(m_1, m_2). \tag{7}$$

We defined the conflict measure between one expert i and the other $M-1$ experts by:

$$\mathrm{Conf}(i,\mathscr{E}) = \frac{1}{M-1}\sum_{j=1, i\neq j}^{M} \mathrm{Conf}(i,j), \tag{8}$$

where $\mathscr{E} = \{1,\ldots,M\}$ is the set of experts in conflict with i. Another definition is given by:

$$\mathrm{Conf}(i,M) = d(m_i, \overline{m_M}), \tag{9}$$

where $\overline{m_M}$ is the bba of the artificial expert representing the combined opinions of all the experts in $\mathscr{E}$ except i.

We use the distance defined in [3], which is for us the most appropriate. See [4] for a comparison of distances in the theory of belief functions. This distance is defined for two basic belief assignments $\mathbf{m}_1$ and $\mathbf{m}_2$ on 2^Θ by:

$$d(m_1,m_2) = \sqrt{\frac{1}{2}(\mathbf{m}_1 - \mathbf{m}_2)^T \underline{\underline{D}}(\mathbf{m}_1 - \mathbf{m}_2)}, \tag{10}$$

where $\underline{\underline{D}}$ is an $2^{|\Theta|} \times 2^{|\Theta|}$ matrix based on Jaccard distance whose elements are:

$$D(A,B) = \begin{cases} 1, \text{if} A = B = \emptyset, \\ \\ \dfrac{|A\cap B|}{|A\cup B|}, \forall A,B \in 2^\Theta. \end{cases} \tag{11}$$

This measure is called a *total conflict* measure. An interesting property of the total conflict is given by $\mathrm{Conf}(m,m) = 0$. That means that there is no conflict between a source and itself (that is not a contradiction). However, we generally do not have $\mathrm{Conf}(m, m_\Theta) = 0$, where $m_\Theta(\Theta) = 1$ is the ignorance.

3 Towards efficient conflict measures

We have seen that we cannot use the mass on the empty set as a conflict measure because of the non-idempotence of the conjunctive rule. We also have seen that the conflict measure based on the distance is not null in general for the ignorance mass. The conjunctive rule does not transfer mass on the empty set if the mass functions are *included*.

Definition *We say that the mass function m_1 is* included *in m_2 if all the focal elements of m_1 are included in all focal elements of m_2. We note this inclusion by $m_1 \subseteq m_2$. The mass functions are* included *if m_1 is included in m_2 or m_2 is included in m_1.*

Therefore these two conflict measures have not an intuitive and expected behavior. Hereafter, we define a new conflict measure having expected properties presented in the following axioms.

Axioms

Let note $\mathrm{Conf}(m_1, m_2)$ a conflict measure between the mass functions m_1 and m_2. We present hereafter essential properties that must verify a conflict measure.

1. Non-negativity: $\mathrm{Conf}(m_1, m_2) \geq 0$
   A negative conflict does not make sens. This axiom is for us necessary.
2. Identity: $\mathrm{Conf}(m_1, m_1) = 0$
   Two equal mass functions are not in conflict. This property is not reached by the global conflict, but seems natural.
3. Symmetry: $\mathrm{Conf}(m_1, m_2) = \mathrm{Conf}(m_2, m_1)$
   The conflict measure must be symmetric. We do not see any case where the non-symmetry can make sens.
4. Normalization: $0 \leq \mathrm{Conf}(m_1, m_2) \leq 1$
   This axiom is may be not necessary to define a conflict measure, but the normalization is very useful in many applications of conflict measure.
5. Inclusion: $\mathrm{Conf}(m_1, m_2) = 0$, iif $m_1 \subseteq m_2$ or $m_2 \subseteq m_1$
   This axiom means if the focal elements of two mass functions are not conflicting (the intersection is never empty), the mass functions are not in conflict and the mass functions cannot be in conflict if they are included. This property is not reached by a distance based conflict measure.

If a conflict measure verifies these axioms that is not necessary a distance. Indeed, we only impose the identity and not the definiteness ($\mathrm{Conf}(m_1, m_2) = 0 \Leftrightarrow m_1 = m_2$). The axiom of inclusion is less restrictive and make more sens for a conflict measure. Moreover, we do not impose the triangle inequality ($\mathrm{Conf}(m_1, m_2) \leq \mathrm{Conf}(m_1, m_3) + \mathrm{Conf}(m_3, m_2)$). It can be interesting to have $\mathrm{Conf}(m_1, m_2) \geq \mathrm{Conf}(m_1, m_3) + \mathrm{Conf}(m_3, m_2)$ meaning that an expert given the mass function m_3 can reduce the conflict. He reach a kind of consensus. Therefore, a distance cannot be used directly to define a conflict measure as before.

Degree of inclusion

We see that the axiom of inclusion seems very important to define a conflict measure. This is the reason why we define here a degree of inclusion measuring how two mass functions are included. Let the inclusion index: $Inc(X_1,Y_2) = 1$ if $X_1 \subseteq Y_2$ and 0 otherwise, where X_1 and Y_2 are two focal elements of m_1 and m_2 respectively. Let $d_{inc}(m_1,m_2)$ a degree of inclusion of m_1 **in** m_2. We can define it by:

$$d_{inc}(m_1,m_2) = \frac{1}{|\mathscr{F}_1||\mathscr{F}_2|} \sum_{X_1 \in \mathscr{F}_1} \sum_{Y_2 \in \mathscr{F}_2} Inc(X_1,Y_2) \qquad (12)$$

where $\mathscr{F}_1$ and $\mathscr{F}_2$ are the set of focal elements of m_1 and m_2 respectively, and $|\mathscr{F}_1|$, $|\mathscr{F}_2|$ are the number of focal elements of m_1 and m_2.

Let $\delta_{inc}(m_1,m_2)$ a degree of inclusion of m_1 **and** m_2 define by:

$$\delta_{inc}(m_1,m_2) = \max(d_{inc}(m_1,m_2), d_{inc}(m_2,m_1)) \qquad (13)$$

This degree gives the maximum of the proportion of focal elements from one mass function included in another one. Therefore, $\delta_{inc}(m_1,m_2) = 1$ if and only if m_1 and m_2 are included, and the axiom of inclusion is reached for $1 - \delta_{inc}(m_1,m_2)$.

A conflict measure

We define a conflict measure between two mass functions m_1 and m_2 by:

$$\mathrm{Conf}(m_1,m_2) = (1 - \delta_{inc}(m_1,m_2))d(m_1,m_2) \qquad (14)$$

where d is the distance defined by the equation (10). All the previous axioms are reached. Indeed the axiom of inclusion is reached by $1 - \delta_{inc}(m_1,m_2)$ and the distance d verify the other axioms. Moreover $0 \leq \delta_{inc}(m_1,m_2) \leq 1$, by the product of $1 - \delta_{inc}$ and d, all the axioms are verified.

For more than two mass functions, the conflict measure between one expert i and the other $M-1$ experts can be defined from the equations (8) or (9).

4 Illustration

Comportment of the proposed conflict measure

We can first note $\mathrm{Conf}(m_1,m_1) = 0$ and $\mathrm{Conf}(m_1,m_\Theta) = 0$ as expected. We have even: if m_1 and m_2 are included then $\mathrm{Conf}(m_1,m_2) = 0$, because the degree of inclusion gives the axiom of inclusion. For example, let's consider:

$$m_1(\theta_1) = m_1(\theta_2) = m_1(\theta_1 \cup \theta_2) = 1/3, \tag{15}$$

$$m_2(\theta_1 \cup \theta_2) = m_2(\theta_1 \cup \theta_2 \cup \theta_3) = 1/2. \tag{16}$$

On this example, $d(m_1,m_2) = 0.3727$. $d_{inc}(m_1,m_2) = 1$ and $d_{inc}(m_2,m_1) = 0.17$, therefore $\delta_{inc}(m_1,m_2) = 1$ and $\mathrm{Conf}(m_1,m_2) = 0$

Note we have $d_{inc}(m_1,m_1) = 0.56$ and $d_{inc}(m_2,m_2) = 0.75$, we only have $d_{inc}(m,m) = 1$ if m is categoric ($m(X) = 1$, $X \in 2^\Theta$).

To illustrate the comportment of the proposed conflict measure we consider:

$$m_3(\theta_3) = m_3(\theta_1 \cup \theta_2 \cup \theta_3) = 0.5. \tag{17}$$

We have $d_{inc}(m_1,m_3) = d_{inc}(m_2,m_3) = 0.5$, but $\mathrm{Conf}(m_1,m_3) = 0.3815$ and $\mathrm{Conf}(m_2,m_3) = 0.3571$. Hence, we obtain:
$\mathrm{Conf}(m_1,m_3) \geq \mathrm{Conf}(m_1,m_2) + \mathrm{Conf}(m_2,m_3)$. m_2 reduce the conflict between m_1 and m_3. If we consider two categorical mass functions such as $m_4(\theta_1) = 1$, $m_5(\theta_2) = 1$ we obtain the maximum of the conflict measure: $\mathrm{Conf}(m_4,m_5) = 1$. That means the most conflicting mass functions are two different categorical mass functions.

On the use of conflict measures

The role of conflict is essential in information fusion. Different ways can be use to manage and reduce the conflict. The conflict can come from the low reliability of the sources. Therefore, we can use this conflict to estimate the reliability of the sources if we cannot learn it on databases as proposed in [10]. Hence, we reduce the conflict before the combination, but we can also directly manage the conflict in the rule of combination as generally made in the theory of belief functions such as explained in [16, 9]. The proposed conflict measure could also use to define combination rules.

According to the application, we do not search always to reduce the conflict. For example, we can use the conflict measure such as an indicator for example in databases [1]. Conflict information can also be an interesting information in some applications such as presented in [12].

5 Conclusion

We propose in this paper an analysis of existing conflict measure. On the base of the drawbacks of these measures, we propose a conflict measure in order to outperform existing ones. This measure is based on the definition of a degree of inclusion. This degree is introduced here in order to quantify how the focal elements of two mass functions are included together. Indeed, we can consider that two mass functions are not in conflict if its are included. The proposed conflict measure, based on five axioms, is then the product of this degree of inclusion and a distance between two mass functions. We see for example this conflict measure can be use to reduce the conflict before or in the combination or as enrichment in databases.

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