A new generalization of the proportional conflict redistribution rule stable in terms of decision

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Abstract

In this chapter, we present and discuss a new generalized proportional conflict redistribution rule. The Dezert-Smarandache extension of the Demster-Shafer theory has relaunched the studies on the combination rules especially for the management of the conflict. Many combination rules have been proposed in the last few years. We study here different combination rules and compare them in terms of decision on didactic example and on generated data. Indeed, in real applications, we need a reliable decision and it is the final results that matter. This chapter shows that a fine proportional conflict redistribution rule must be preferred for the combination in the belief function theory.

Keywords: Experts fusion, DST, DSmT, generalized P R, DSmH.

1 Introduction

Many fusion theories have been studied for the combination of the experts opinions such as voting rules [24, 10], possibility theory [26, 6], and belief function theory [2, 14]. We can divide all these fusion approaches into four steps: the modelization, the parameters estimation depending on the model (not always necessary), the combination, and the decision. The most difficult step is presumably the first one. If both possibility and probability-based theories can modelize imprecise and uncertain data at the same time, in a lot of applications, experts can express their certitude on their perception of the reality. As a result, probabilities theory such as the belief function theory is more adapted. In the context of the belief function theory, the Dempster-Shafer theory (DST) [2, 14] is based on the use of functions defined on the power set 2^Θ (that is the set of all the disjunctions of the elements of Θ). Hence the experts can express their opinion not only on Θ but also on 2^Θ as in the probabilities theory. The extension of this power set into the hyper power set D^Θ (that is the set of all the disjunctions and conjunctions of the elements of Θ) proposed by Dezert and Smarandache [3], gives more freedom to the expert. This extension of the DST is called Dezert-Smarandache Theory (DSmT).

This extension has relaunched the studies on the combination rules. The combination of multiple sources of information has still been an important subject of research since the proposed combination rule given by Dempster [2]. Hence, many solutions have been studied in order to manage the conflict [25, 7, 9, 21, 22, 11, 12, 18, 8]. These combination rules are the most of time compared following the properties of the operator such as associativity, commutativity, linearity, anonymity and on special and simple cases of experts responses [21, 23, 1].

In real applications, we need a reliable decision and it is the final results that matter. Hence, for a given application, the best combination rule is the rule given the best results. For the decision step, different functions such as credibility, plausibility and pignistic probability [14, 19, 5] are usually used.

In this chapter, we discuss and compare different combination rules especially managing the conflict. First, the principles of the DST and DSmT are recalled. We present the formalization of the belief function models, different rules of combination and decision. One the combination rule (PCR5) proposed by [18] for two experts is mathematically one of the best for the proportional redistribution of the conflict applicable in the context of the DST and the DSmT. In the section 3, we propose a new extension of this rule for more experts, the PCR6 rule. This new rule is compared to the generalized PCR5 rule given in [4], in the section 4. Then this section presents a comparison of different combination rules in terms of decision in a general case, where the experts opinions are randomly simulated. We demonstrate also that some combination rules are different in terms of decision, in the case of two experts and two classes.

2 Theory Bases

2.1 Belief Function Models

The belief functions or basic belief assignments m are defined by the mapping of the power set 2^Θ onto $[0,1]$, in the DST, and by the mapping of the hyper-power set D^Θ onto $[0,1]$, in the DSmT, with:

$$m(\emptyset) = 0, \tag{1}$$

and

$$\sum_{X \in 2^\Theta} m(X) = 1, \tag{2}$$

in the DST, and

$$\sum_{X \in D^\Theta} m(X) = 1, \tag{3}$$

in the DSmT.

The equation (1) is the hypothesis at a closed world [14, 15]. We can define the belief function only with:

$$m(\emptyset) > 0, \tag{4}$$

and the world is open [19]. In a closed world, we can also add one element in order to propose an open world.

These simple conditions in equation (1) and (2) or (1) and (3), give a large panel of definitions of the belief functions, which is one the difficulties of the theory. The belief functions must therefore be chosen according to the intended application.

2.2 Combination rules

Many combination rules have been proposed in the last few years in the context of the belief function theory ([25, 7, 19, 21, 15, 18], *etc.*). In the context of the DST, the combination rule most used today seems to be the conjunctive rule given by [19] for all $X \in 2^\Theta$ by:

$$m_c(X) = \sum_{Y_1 \cap ... \cap Y_M = X} \prod_{j=1}^{M} m_j(Y_j), \tag{5}$$

where $Y_j \in 2^\Theta$ is the response of the expert j, and $m_j(Y_j)$ the associated belief function.

However, the conflict can be redistributed on partial ignorance like in the Dubois and Prade rule [7], a mixed conjunctive and disjunctive rule given for all $X \in 2^\Theta$, $X \neq \emptyset$ by:

$$m_{\mathrm{DP}}(X) = \sum_{Y_1 \cap ... \cap Y_M = X} \prod_{j=1}^{M} m_j(Y_j) + \sum_{\substack{Y_1 \cup ... \cup Y_M = X \\ Y_1 \cap ... \cap Y_M = \emptyset}} \prod_{j=1}^{M} m_j(Y_j), \tag{6}$$

where $Y_j \in 2^\Theta$ is the response of the expert j, and $m_j(Y_j)$ the associated belief function.

In the context of the DSmT, the conjunctive rule can be used for all $X \in D^\Theta$ and $Y \in D^\Theta$. The rule given by the equation (6), called DSmH [15], can be write in D^Θ for all $X \in D^\Theta$, $X \not\equiv \emptyset$ [1] by:

$$\begin{aligned} m_H(X) = & \sum_{Y_1 \cap ... \cap Y_M = X} \prod_{j=1}^{M} m_j(Y_j) + \sum_{\substack{Y_1 \cup ... \cup Y_M = X \\ Y_1 \cap ... \cap Y_M \equiv \emptyset}} \prod_{j=1}^{M} m_j(Y_j) + \\ & \sum_{\substack{\{u(Y_1 \cup ... \cup u(Y_M = X\} \\ Y_1, ..., Y_M \equiv \emptyset}} \prod_{j=1}^{M} m_j(Y_j) + \sum_{\substack{\{u(Y_1 \cup ... \cup u(Y_M \equiv \emptyset \text{ and } X = \Theta\} \\ Y_1, ..., Y_M \equiv \emptyset}} \prod_{j=1}^{M} m_j(Y_j), \end{aligned} \tag{7}$$

where $Y_j \in D^\Theta$ is the response of the expert j, $m_j(Y_j)$ the associated belief function, and $u(Y)$ is the function giving the union that compose Y [16]. For example if $Y = (A \cap B) \cup (A \cap C)$, $u(Y) = A \cup B \cup C$.

If we want to take the decision only on the elements in Θ, some rules propose to redistribute the conflict on these elements. The most accomplished is the PCR5 given in [18] for two experts and for $X \in D^\Theta$, $X \neq \emptyset$ by:

$$\begin{aligned} m_{PCR5}(X) = & \, m_c(X) + \\ & \sum_{\substack{Y \in D^\Theta, \\ X \cap Y \equiv \emptyset}} \left(\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)} \right), \end{aligned} \tag{8}$$

[1] The notation $X \not\equiv \emptyset$ means that $X \neq \emptyset$ and following the chosen model in D^Θ, X is not one of the element of D^Θ defined as $\emptyset$. For example, if $\Theta = \{A, B, C\}$, we can define a model for which the expert can provide a mass on $A \cap B$ and not on $A \cap C$, so $A \cap B \neq \emptyset$ and $A \cap B = \emptyset$

where $m_c(.)$ is the conjunctive rule given by the equation (5).

Note that more rules managing the conflict have been proposed [25, 9, 11, 12, 18, 8].

The comparison of all the combination rules is not the scope of this paper.

2.3 Decision rules

The decision is a difficult task. No measures are able to provide the best decision in all the cases. Generally, we consider the maximum of one of the three functions: credibility, plausibility, and pignistic probability.

In the context of the DST, the credibility function is given for all $X \in 2^\Theta$ by:

$$\text{bel}(X) = \sum_{Y \in 2^X, Y \neq \emptyset} m(Y). \tag{9}$$

The plausibility function is given for all $X \in 2^\Theta$ by:

$$\text{pl}(X) = \sum_{Y \in 2^\Theta, Y \cap X \neq \emptyset} m(Y) = \text{bel}(\Theta) - \text{bel}(X^c), \tag{10}$$

where X^c is the complementary of X. The pignistic probability, introduced by [20], is here given for all $X \in 2^\Theta$, with $X \neq \emptyset$ by:

$$\text{betP}(X) = \sum_{Y \in 2^\Theta, Y \neq \emptyset} \frac{|X \cap Y|}{|Y|} \frac{m(Y)}{1 - m(\emptyset)}. \tag{11}$$

Generally the maximum of these functions is taken on the elements in Θ, but we will give the values on all the focal elements.

In the context of the DSmT the corresponding generalized functions have been proposed [5, 15]. The generalized credibility Bel is defined by:

$$\text{Bel}(X) = \sum_{Y \in D^\Theta, Y \subseteq X, Y \not\equiv \emptyset} m(Y) \tag{12}$$

The generalized plausibility Pl is defined by:

$$\text{Pl}(X) = \sum_{Y \in D^\Theta, X \cap Y \not\equiv \emptyset} m(Y) \tag{13}$$

The generalized pignistic probability is given for all $X \in D^\Theta$, with $X \neq \emptyset$ is defined by:

$$\text{GPT}(X) = \sum_{Y \in D^\Theta, Y \not\equiv \emptyset} \frac{\mathcal{C}_\mathcal{M}(X \cap Y)}{\mathcal{C}_\mathcal{M}(Y)} m(Y), \tag{14}$$

where $\mathcal{C}_\mathcal{M}(X)$ is the DSm cardinality corresponding to the number of parts of X in the Venn diagram of the problem [5, 15].

If the credibility function provides a pessimist decision, the plausibility function is often too optimist. The pignistic probability is often taken as a compromise. We present the three functions for our models.

3 The generalized PCR rules

In the equation (8), the PCR5 is given for two experts only. Two extensions for three experts and two classes are given in [17], and the equation for M experts, for $X \in D^\Theta$, $X \not\equiv \emptyset$ is given in [4] by:

$$m_{\mathrm{PCR5}}(X) = m_{\mathrm{c}}(X) + \sum_{i=1}^{M} m_i(X) \sum_{\substack{(Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1} \\ \bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset}} \frac{\left(\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \mathbb{1}_{j>i}\right) \prod_{Y_{\sigma_i(j)}=X} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{\sum_{Z \in \{X, Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}\}} \prod_{Y_{\sigma_i(j)}=Z} \left(m_{\sigma_i(j)}(Y_{\sigma_i(j)}).T(X{=}Z, m_i(X))\right)}, \quad (15)$$

where σ_i counts from 1 to M avoiding i:

$$\begin{cases} \sigma_i(j) = j & \text{if } j < i, \\ \sigma_i(j) = j+1 & \text{if } j \geq i, \end{cases} \quad (16)$$

and:

$$\begin{cases} T(B,x) = x & \text{if } B \text{ is true}, \\ T(B,x) = 1 & \text{if } B \text{ is false}. \end{cases} \quad (17)$$

We propose another generalization of the equation (8) for M experts, for $X \in D^\Theta$, $X \neq \emptyset$:

$$m_{\mathrm{PCR6}}(X) = m_{\mathrm{c}}(X) + \sum_{i=1}^{M} m_i(X)^2 \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1}}} \left(\frac{\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X) + \sum_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right), \quad (18)$$

where σ is defined like in (16).

$m_i(X) + \sum_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \neq 0$, m_c is the conjunctive consensus rule given by the equation (5).

We can propose two more generalized rules given by:

$$m_{\mathrm{PCR6f}}(X) = m_{\mathrm{c}}(X) + \sum_{i=1}^{M} m_i(X) f(m_i(X)) \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1}}} \left(\frac{\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{f(m_i(X)) + \sum_{j=1}^{M-1} f(m_{\sigma_i(j)}(Y_{\sigma_i(j)}))} \right), \quad (19)$$

with the same notations that in the equation (18), and f an increasing function defined by the mapping of $[0,1]$ onto $\mathbb{R}^+$.

The second generalized rule is given by:

$$m_{\mathrm{PCR6g}}(X) = m_{\mathrm{c}}(X) + \sum_{i=1}^{M} \sum_{\substack{\overset{M-1}{\underset{k=1}{\cap}} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (D^{\Theta})^{M-1}}}$$

$$m_i(X) \frac{\left(\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})\right)\left(\prod_{Y_{\sigma_i(j)}=X} 1\!\!1_{j>i}\right) g\left(m_i(X) + \sum_{Y_{\sigma_i(j)}=X} m_{\sigma_i(j)}(Y_{\sigma_i(j)})\right)}{\sum_{Z\in\{X,Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}\}} g\left(\sum_{Y_{\sigma_i(j)}=Z} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) + m_i(X) 1\!\!1_{X=Z}\right)}, \tag{20}$$

with the same notations that in the equation (18), and g an increasing function defined by the mapping of $[0,1]$ onto $\mathbb{R}^+$.

For instance, we can choose $f(x) = g(x) = x^{\alpha}$, with $\alpha \in \mathbb{R}^+$.

Algorithms for the Dubois and Prade (equation (6)), the PCR5 (equation (15)), the PCR6 (equation (18)), the PCR6f (equation (19)), and the PCR6g (equation (20)) combinations are given in appendix.

Rem rks on the gener lized PCR rules

- $\overset{M-1}{\underset{k=1}{\cap}} Y_k \cap X \equiv \emptyset$ means that $\overset{M-1}{\underset{k=1}{\cap}} Y_k \cap X$ is considered as a conflict by the model: $m_i(X) \prod_{k=1}^{M-1} m_{\sigma_i(k)}(Y_{\sigma_i(k)})$ has to be redistributed on X and the Y_k.

- The second term of the equation (18) is null if $\overset{M-1}{\underset{k=1}{\cap}} Y_k \cap X \not\equiv \emptyset$, hence in a general model in D^{Θ} for all X and $Y \in D^{\Theta}$, $X \cap Y \neq \emptyset$. The PCR5 and PCR6 are exactly the conjunctive rule: there is never any conflict. However in $2^{2^{\Theta}}$, $\exists X, Y \in 2^{2^{\Theta}}$ such as $X \cap Y = \emptyset$.

- One of the principal problem of the PCR5 and PCR6 rules is the non associativity. That is is a real problem for dynamic fusion. Take for example three experts and two classes giving:

	$\emptyset$	A	B	Θ
Expert 1	0	1	0	0
Expert 2	0	0	1	0
Expert 3	0	0	1	0

If we fuse the expert 1 and 2 and then 3, the PCR5 and the PCR6 rules give:

$$m_{12}(A) = 0.5, \quad m_{12}(B) = 0.5,$$

and

$$m_{(12\ 3}(A) = 0.25, \quad m_{(12\ 3}(B) = 0.75.$$

Now if we fuse the expert 2 and 3 and then 1, the PCR5 and the PCR6 rules give:

$$m_{23}(A) = 0, \quad m_{23}(B) = 1,$$

and

$$m_{(12\ 3}(A) = 0.5, \quad m_{(12\ 3}(B) = 0.5,$$

and the result is not the same.

With the generalized PCR6 rule we obtain:

$$m_{(123}\ (A) = 1/3, \quad m_{(123}\ (B) = 2/3,$$

a more intuitive result.

- The conflict is not only redistributed on singletons. For example if three experts give:

	$A \cup B$	$B \cup C$	$A \cup C$	Θ
Expert 1	0.7	0	0	0.3
Expert 2	0	0	0.6	0.4
Expert 3	0	0.5	0	0.5

The conflict is given here by 0.7×0.6×0.5=0.21, with the generalized PCR6 rule we obtain:

$$m_{(123}\ (A) = 0.21,$$
$$m_{(123}\ (B) = 0.14,$$
$$m_{(123}\ (C) = 0.09,$$
$$m_{(123}\ (A \cup B) = 0.14 + 0.21.\frac{7}{18} \simeq 0.2217,$$
$$m_{(123}\ (B \cup C) = 0.06 + 0.21.\frac{5}{18} \simeq 0.1183,$$
$$m_{(123}\ (A \cup C) = 0.09 + 0.21.\frac{6}{18} = 0.16,$$
$$m_{(123}\ (\Theta) = 0.06.$$

4 Discussion on the decision following the combination rules

In order to compare the previous rules in this section, we study the decision on the basic belief assignments obtained by the combination. Hence, we consider here the induced order on the singleton given by the plausibility, credibility, pignistic probability functions, or directly by the masses. Indeed, in order to compare the combination rules, we think that the study on the induced order of these functions is more informative than the obtained masses values. All the combination rules presented here are not idempotent, for instance for the conjunctive non-normalized rule:

	$\emptyset$	A	B	C
m_1	0	0.6	0.3	0.1
m_1	0	0.6	0.3	0.1
m_{11}	0.54	0.36	0.09	0.01

So, if we only compare the rules looking the obtained masses, we have normalize them with the auto-conflict given by the combination of a mass with itself. However, if $m_1(A) > m_1(B)$, then $m_{11}(A) > m_{11}(B)$.

4.1 Extending the PCR rule for more than two experts

In [17], two approaches are presented in order to extend the PCR5 rule. The second approach suggests to fuse the first two experts and then fuse the third expert. However the solution depend on the order of the experts because of the non-associativity of the rule, and so it is not satisfying.

The first approach proposed in [17], that is the equation (15) proposes to redistribute the conflict about the singleton, *e.g.* if we have $m_1(A)m_3(B)m_2(A \cup B)$, the conflict is redistributed on A and B proportionally to $m_1(A)$ and $m_3(B)$. But this approach do not give solution if we have for instance $m_1(A \cup B)m_2(B \cup C)m_3(A \cup C)$ where the conflict is $A \cap B \cap C$ and we have no idea on the masses for A, B and C.

Moreover, if we have $m_(A)m_2(B)m_3(B)$ the proposed solution distribute the conflict to A and B with respect to $m_1(A)$ and $m_2(B)m_3(B)$ and not $m_2(B) + m_3(B)$ that is more intuitive. For example, if $m_1(A) = m_2(B) = m_3(B) = 0.5$, 0.0833 and 0.0416 is added to the masses A and B respectively, while there is more consensus on B than on A and we would expected the contrary: 0.0416 and 0.0833 could be added to the masses A and B respectively.

What is more surprising are the results given by PCR5 and PCR6 on the following example:

	A	B	C	D	E	F	G
Expert 1	0.0	0.57	0.43	0.0	0.0	0.0	0.0
Expert 2	0.58	0.0	0.0	0.42	0.0	0.0	0.0
Expert 3	0.58	0.0	0.0	0.0	0.42	0.0	0.0
Expert 4	0.58	0.0	0.0	0.0	0.0	0.42	0.0
Expert 5	0.58	0.0	0.0	0.0	0.0	0.0	0.42

As all the masses are on singletons, neither PCR5 nor PCR6 can put any mass on total or partial ignorance. So the fusion result is always a probability, and $\text{bel}(X) = \text{betP}(X) = \text{pl}(X)$.

Conflict is total: conjunctive rule does not provide any information. PCR5 and PCR6 give the following results:

	A	B	C	D	E	F	G
PCR5	0.1915	0.2376	0.1542	0.1042	0.1042	0.1042	0.1042
PCR6	0.5138	0.1244	0.0748	0.0718	0.0718	0.0718	0.0718

So decision is “A” according to PCR6, and decision is “B” according to PCR5. However, for any subset of 2, 3 or 4 experts, decision is “A” for any of these combination rules.

4.2 Stability of decision process

The space where experts can define their opinions on which n classes are present in a given tile is a part of $[0,1]^n$: $\mathcal{E} = [0,1]^n \cap \left\{ (x_1, ..., x_n) \in \mathrm{I\!R} / \sum_{i=1}^{n} x_i \leq 1 \right\}$. In order to study the different combination rules, and the situations where they differ, we use a Monte Carlo method, considering the masses given on each class (a_X) by each expert, as uniform variables, filtering them by the condition $\sum_{X \in \Theta} a_X \leq 1$ for one expert.

Thus, we measure the proportion of situations where decision differs between the conjunctive combination rule, and the PCR, where conflict is proportionally distributed.

We can not choose $A \cap B$, as the measure of $A \cap B$ is always lower (or equal with probability 0) than the measure of A or B. In the case of two classes, $A \cup B$ is the ignorance, and is usually excluded (as it always maximizes bel, pl, betP, Bel, Pl and GPT). We restrict the possible choices to singletons, A, B, etc. Therefore, it is equivalent to tag the tile by the most credible class (maximal for bel), the most plausible (maximal for pl), the most probable (maximal for betP) or the heaviest (maximal for m), as the only focal elements are singletons, Θ and $\emptyset$.

The only situation where the total order induced by the masses m on singletons can be modified is when the conflict is distributed on the singletons, as is the case in the PCR method.

Thus, for different numbers of classes, the decision obtained by fusing the experts' opinions is much less stable:

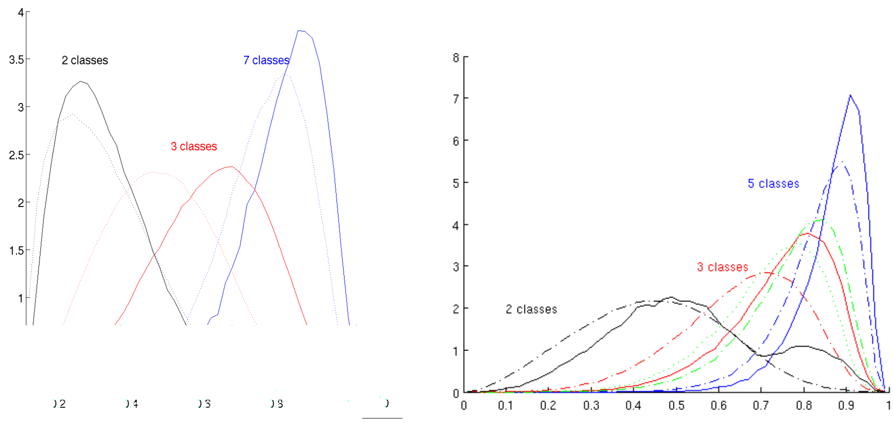
number of classes	2	3	4	5	6	7
decision change in the two experts case						
PCR/DST	0.61%	5.51%	9.13%	12.11%	14.55%	16.7%
PCR/DP	0.61%	2.25%	3.42%	4.35%	5.05%	5.7%
DP/DST	0.00%	3.56%	6.19%	8.39%	10.26%	11.9%
decision change in the three experts case						
PCR6/DST	1.04%	8.34%	13.90%	18.38%	21.98%	25.1%
PCR6/DP	1.04%	5.11%	7.54%	9.23%	10.42%	11.3%
DP/DST	0.00%	4.48%	8.88%	12.88%	16.18%	19.0%

Therefore, the specificity of PCR6 appears mostly with more than two classes, and the different combination rules are nearly equivalent when decision must be taken within two possible classes.

For two experts and two classes, the mixed rule (DP) and the conjunctive rule (DST) are equivalent. For three experts, we use the generalized PCR6 (18).

The percentage of decision differences defines a distance between fusion methods: $d(\text{PCR6,DST}) \leq d(\text{PCR6,DP}) + d(\text{DP,DST})$. The two other triangular inequalities are also true. As we have $d(PCR6, DST) \geq d(\text{PCR6,DP})$ and $d(\text{PCR,DST}) \geq d(\text{DP,DST})$ for any number of experts or classes, we can conclude that the mixed rule lies between the PCR6 method and the conjunctive rule.

The figure 1 shows the density of conflict within $\mathcal{E}$. The left part shows the conflict for two random experts and a number of classes of 2, 3 or 7. Plain lines



2 classes
7 classes
3 classes
2 classes
3 classes
5 classes

$$m_c(B) = b_1 + b_2 - b_1b_2 - a_1b_2 - a_2b_1 = b_1 + b_2 - b_1b_2 - m_c(\emptyset),$$

$$m_c(\Theta) = (1 - a_1 - b_1)(1 - a_2 - b_2).$$

PCR gives:

$$m_{PCR}(A) = m(A) + \frac{a_1^2 b_2}{a_1 + b_2} + \frac{a_2^2 b_1}{a_2 + b_1},$$

$$m_{PCR}(B) = m(B) + \frac{a_1 b_2^2}{a_1 + b_2} + \frac{a_2 b_1^2}{a_2 + b_1},$$

$$m_{PCR}(\emptyset) = 0 \quad \text{and} \quad m_{PCR}(\Theta) = m_c(\Theta).$$

The stability of the decision is reached if we do not have:

$$\left\{\begin{array}{l} m_c(A) > m_c(B) \text{ and } m_{PCR}(A) < m_{PCR}(B) \\ \text{or} \\ m_c(A) < m_c(B) \text{ and } m_{PCR}(A) > m_{PCR}(B) \end{array}\right. \tag{21}$$

That means for all a_1, a_2, b_1 and $b_2 \in [0, 1]$:

$$\begin{array}{l} \left\{\begin{array}{l} a_2 + a_1(1-a_2) - b_1(b_2 - 1) - b_2 > 0 \\ a_1(1-a_2) + a_2\left(\left(1 + b_1\left(1 - \frac{2}{(1+a_2/b_1}\right)\right) - b_1(1-b_2)\right. \\ \qquad\qquad - b_2\left(1 + a_1\left((1 - \frac{2}{(1+b_2/a_1}\right)\right) < 0 \\ a_1 + b_1 \in [0,1] \\ a_2 + b_2 \in [0,1] \end{array}\right. \\ \text{or} \\ \left\{\begin{array}{l} a_2 + a_1(1-a_2) - b_1(b_2 - 1) - b_2 < 0 \\ a_1(1-a_2) + a_2\left(\left(1 + b_1\left(1 - \frac{2}{(1+a_2/b_1}\right)\right) - b_1(1-b_2)\right. \\ \qquad\qquad - b_2\left(1 + a_1\left((1 - \frac{2}{(1+b_2/a_1}\right)\right) > 0 \\ a_1 + b_1 \in [0,1] \\ a_2 + b_2 \in [0,1] \end{array}\right. \end{array} \tag{22}$$

This system of inequation is difficult to solve, but with the help of a Monte Carlo method, considering the weights a_1, a_2, b_1 and b_2, as uniform variables we can estimate the proportion of points (a_1, a_2, b_1, b_2) solving this system.

We note that absence of solution in spaces where $a_1 + b_1 > 1$ or $a_2 + b_2 > 1$ comes from the two last conditions of the system. Also there is no solution if $a_1 = b_1$ (or $a_2 = b_2$ by symmetry) and if $a_1 = b_2$ (or $a_2 = b_1$ by symmetry). This is proved analytically.

4.3.1 C se $a_1 = b_1$

In this situation, expert 1 considers that the data unit is equally filled with classes A and B:

	$\emptyset$	A	B	Θ
Expert 1	0	x	x	$1 - 2x$
Expert 2	0	y	z	$1 - y - z$

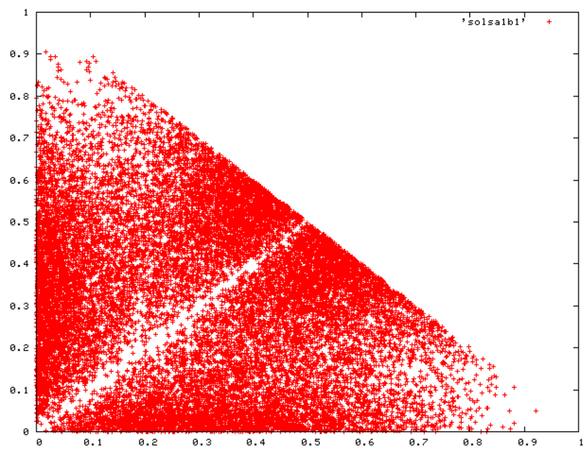


Figure 2: Decision changes, projected on the plane a_1, b_1.

The conjunctive rule yields:

$$m_c(\emptyset) = 2xy,$$

$$m_c(A) = x + y - 2xy - xz = x - m_c(\emptyset) + y(1 - x),$$
$$m_c(B) = x + y - xy - 2xz = x - m_c(\emptyset) + z(1 - x),$$
$$m_c(\Theta) = 1 - 2x - y - z + 2xy + 2xz.$$

Therefore, as $1 - x \geq 0$:

$$m_c(A) > m_c(B) \iff y > z.$$

The PCR yields:

$$m_{PCR}(\emptyset) = 0$$

$$m_{PCR}(A) = x - m_c(\emptyset) + y(1 - x) + \frac{x^2 z}{x + z} + \frac{xy^2}{x + y},$$

$$m_{PCR}(B) = x - m_c(\emptyset) + z(1 - x) + \frac{xz^2}{x + z} + \frac{x^2 y}{x + y},$$

$$m_{PCR}(\Theta) = 1 - 2x - y - z + 2xy + 2xz.$$

So, we have:

$$\begin{aligned}(m_{PCR}(A) + m_c(\emptyset) - x)(x + y)(x + z) &= y(1 - x)(x + z)(x + y) \\ &\quad + x^2 z(x + y) + y^2 x(x + z) \\ &= y(x + y)(x + z) + x^3(z - y)\end{aligned}$$

$$(m_{PCR}(B) + m_c(\emptyset) - x)(x + y)(x + z) = z(x + y)(x + z) - x^3(z - y),$$

$$m_{PCR}(A) > m_{PCR}(B) \iff (y - z)((x + y)(x + y) - 2x^3) > 0.$$

As $0 \leq x \leq \frac{1}{2}$, we have $2x^3 \leq x^2 \leq (x + y)(x + z)$. So $m_{PCR}(A) > m_{PCR}(B)$ if and only if $y > z$.

That shows that the stability of the decision is reached if $a_1 = b_1$ for all a_2 and $b_2 \in [0, 1]$ or by symmetry if $a_2 = b_2$ for all a_1 and $b_1 \in [0, 1]$.

4.3.2 C se $a_1 = b_2$

In this situation, expert 1 believes A and the expert 2 believes B with the same weight:

	$\emptyset$	A	B	Θ
Expert 1	0	x	y	$1 - x - y$
Expert 2	0	z	x	$1 - x - z$

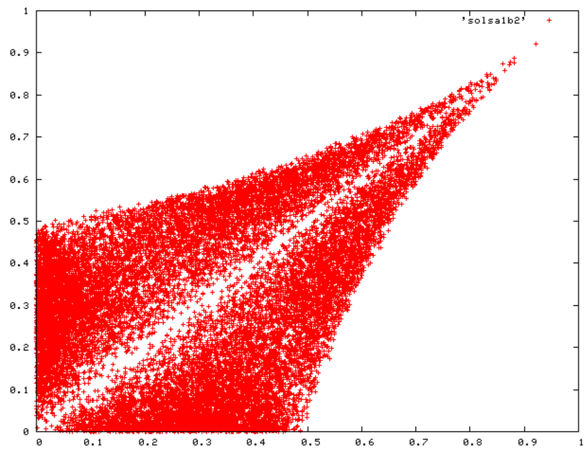


Figure 2: Decision changes, projected on 7.23733(pla]TJ/F89.963Tf7.2371.495Td[(=)]TJ/F119.96

The PCR yields:

$$m_{PCR}(\emptyset) = 0,$$

$$m_{PCR}(A) = x + z - xz - m_c(\emptyset) = -x^2 + x(1-z) + z(1-y) + \frac{x^3}{2x} + \frac{yz^2}{y+z},$$

$$m_{PCR}(B) = x + y - xy - m_c(\emptyset) = -x^2 + x(1-y) + y(1-z) + \frac{x^3}{2x} + \frac{y^2 z}{y+z},$$

$$m_{PCR}(\Theta) = 1 + m_c(\emptyset) - 2x - y - z + x(y+z).$$

Therefore:

$$m_{PCR}(A) > m_{PCR}(B) \iff (y-z)\left((x-1)(y+z) - yz\right) > 0,$$

as $(x-1) \leq 0$, $(x-1)(y+z) - yz \leq 0$ and:

$$m_{PCR}(A) > m_{PCR}(B) \iff y > z.$$

That shows that the stability of the decision is reached if $a_1 = b_2$ for all a_2 and $b_1 \in [0,1]$ or by symmetry if $a_2 = b_1$ for all a_1 and $b_2 \in [0,1]$.

4.3.3 C se $a_2 = 1 - a_1$

We can notice that if $a_1 + a_2 > 1$, no change occurs. In this situation, we have $b_1 + b_2 < 1$, but calculus is still to be done.

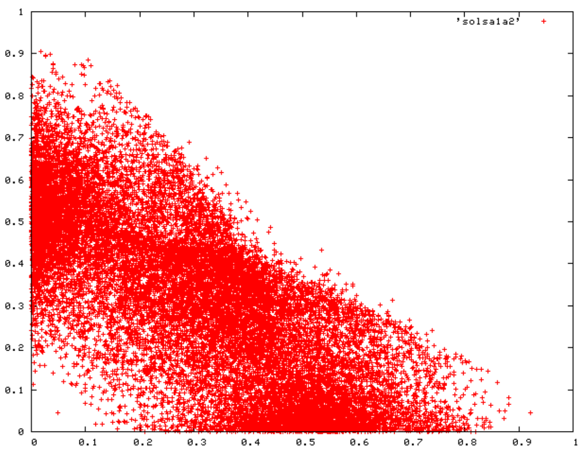


Figure 4: Decision changes, projected on the plane a_1, a_2.

In this situation, if $a_2 = 1 - a_1$:

	$\emptyset$	A	B	Θ
Expert 1	0	x	y	$1-x-y$
Expert 2	0	$1-x$	z	$x-z$

The conjunctive rule yields:

$$m_c(\emptyset) = xz + (1-x)y,$$

$$m_c(A) = 1 + x^2 - x - y + xy - xz,$$

$$m_c(B) = z - yz + xy - xz,$$

$$m_c(\Theta) = -x^2 + x + xz - xy + yz - z.$$

Therefore:

$$m_c(A) > m_c(B) \iff 1 + x^2 - x > y + z - yz,$$

$$\iff x(1-x) > (1-y)(1-z),$$

as $z < x$ and $x < 1-y$, $m_c(A) > m_c(B)$ is always true.

The PCR yields:

$$m_{PCR}(\emptyset) = 0,$$

$$m_{PCR}(A) = m_c(A) + \frac{x^2 z}{x+z} + \frac{(1-x)^2 y}{1-x+y},$$

$$m_{PCR}(B) = m_c(B) + \frac{xz^2}{x+z} + \frac{(1-x)y^2}{1-x+y},$$

$$m_{PCR}(\Theta) = m_c(\Theta).$$

Therefore:

$$m_{PCR}(A) > m_{PCR}(B)$$

is always true.

Indeed we have $m_c(A) > m_c(B)$ is always true and:

$$\frac{x^2 z}{x+z} > \frac{xz^2}{x+z}$$

because $x > z$ and:

$$\frac{(1-x)^2 y}{1-x+y} > \frac{(1-x)y^2}{1-x+y}$$

because $1 - x > y$.

That shows that the stability of the decision is reached if $a_2 = 1 - a_1$ for all a_2 and $a_1 \in [0,1]$ or by symmetry if $a_1 = 1 - a_2$ for all a_1 and $a_2 \in [0,1]$.

5 Conclusion

In this chapter, we have proposed a study of the combination rules compared in term of decision. A new generalized proportional conflict redistribution (PCR6) rule have been proposed and discussed. We have presented the pro and con of this rule. The PCR6 rule is more intuitive than the PCR5. We have shown on randomly generated data, that there is a difference of decision following the choice of the combination rule (for the non-normalized conjunctive rule, the mixed conjunctive and disjunction rule of Dubois and Prade, the PCR5 rule and the PCR6 rule). We have also proven, on a two experts and two classes case, the changes following the values of the basic belief assignments. This difference can be very small in percentage and we can not say on these data if it is a significantly difference. We have conducted this comparison on real data in the chapter [13].

All this discussion comes from a fine proportional conflict distribution initiated by the consideration of the extension of the discernment space in D^Θ. The generalized PCR6 rule can be used on 2^Θ or D^Θ.

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Appendix: Algorithms

An expert e is an association of a list of focal classes and their masses. We write $size(e)$ the number of its focal classes. The focal classes are $e[1]$, $e[2]$, $\ldots$, $e[size(e)]$. The mass associated to a class c is $e(c)$, written with paranthesis.

D t : n experts ex: $ex[1] \ldots ex[n]$
Result: Fusion of ex by Dubois-Prade method : edp
for $i = 1$ *to* n **do**
  fore ch c *in* $ex[i]$ **do**
    Append c to $cl[i]$;
fore ch *ind in* 1, *size*($cl[1]$)] $\times$ 1, *size*($cl[2]$)] $\times$ $\ldots \times$ 1, *size*($cl[n]$)]
do
  $s \leftarrow \Theta$;
  for $i = 1$ *to* n **do**
    $s \leftarrow s \cap cl[i][ind[i]]$;
  if $s = \emptyset$ **then**
    $lconf \leftarrow 1$;
    $u \leftarrow \emptyset$;
    for $i = 1$ *to* n **do**
      $u \leftarrow p \cup cl[i][ind[i]]$;
      $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
    $edp(u) \leftarrow edp(u) + lconf$;
  else
    $lconf \leftarrow 1$;
    for $i = 1$ *to* n **do**
      $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
    $edp(s) \leftarrow edp(s) + lconf$;

```
D t    : n experts ex: ex[1] . . . ex[n]
Result: Fusion of ex by PCR5 method : ep
for i = 1 to n do
    fore ch c in ex[i] do
        Append c to cl[i];
fore ch ind in  1, size(cl[1]))] ×  1, size(cl[2]))] × . . . ×  1, size(cl[n]))]
do
    s ← Θ;
    for i = 1 to n do
        s ← s ∩ cl[i][ind[i]];
    if s = ∅ then
        lconf ← 1;
        el is an empty expert;
        for i = 1 to n do
            lconf ← lconf × ex[i](cl[i][ind[i]]);
            if cl[i][ind[i]] in el then
                el(cl[i][ind[i]]) ← el(cl[i][ind[i]]) ∗ ex[i](cl[i][ind[i]]);
            else
                el(cl[i][ind[i]]) ← ex[i](cl[i][ind[i]]);
        for c in el do
            sum ← sum + el(c);
        for c in el do
            ep(c) ← ep(c) + g(el(c)) ∗ lconf/sum;
    else
        lconf ← 1;
        for i = 1 to n do
            lconf ← lconf × ex[i](cl[i][ind[i]]);
        ep(s) ← ep(s) + lconf;
```

```
D  t    : n experts ex: ex[1] ... ex[n]
Result: Fusion of ex by PCR6 method : ep
for i = 1 to n do
    fore  ch c in ex[i] do
        Append c to cl[i];

fore  ch ind in  1, size(cl[1])] ×  1, size(cl[2])] × ... ×  1, size(cl[n])]
do
    s ← Θ;
    for i = 1 to n do
        s ← s ∩ cl[i][ind[i]];
    if s = ∅ then
        lconf ← 1;
        sum ← 0;
        for i = 1 to n do
            lconf ← lconf × ex[i](cl[i][ind[i]]);
            sum ← sum + ex[i](cl[i][ind[i]]);
        for i = 1 to n do
            ep(ex[i][ind[i]])  ←  ep(ex[i][ind[i]]) + ex[i](cl[i][ind[i]]) *
            lconf/sum;
    else
        lconf ← 1;
        for i = 1 to n do
            lconf ← lconf × ex[i](cl[i][ind[i]]);
        ep(s) ← ep(s) + lconf;
```

D t : n experts ex: $ex[1] \dots ex[n]$

D t : A non-decreasing positive function f

Result: Fusion of ex by PCR6$_f$ method : ep

```
for i = 1 to n do
    fore ch c in ex[i] do
        Append c to cl[i];

fore ch ind in 1, size(cl[1])] × 1, size(cl[2])] × ... × 1, size(cl[n])]
do
    s ← Θ;
    for i = 1 to n do
        s ← s ∩ cl[i][ind[i]];
    if s = ∅ then
        lconf ← 1;
        sum ← 0;
        for i = 1 to n do
            lconf ← lconf × ex[i](cl[i][ind[i]]);
            sum ← sum + f(ex[i](cl[i][ind[i]]));
        for i = 1 to n do
            ep(ex[i][ind[i]]) ← ep(ex[i][ind[i]]) + f(ex[i](cl[i][ind[i]])) *
            lconf/sum;
    else
        lconf ← 1;
        for i = 1 to n do
            lconf ← lconf × ex[i](cl[i][ind[i]]);
        ep(s) ← ep(s) + lconf;
```

D t : n experts ex: $ex[1] \ldots ex[n]$
D t : A non-decreasing positive function g
Result: Fusion of ex by PCR6_g method : ep

for i = *1 to n* **do**
    fore ch c *in* $ex[i]$ **do**
        Append c to $cl[i]$;

fore ch *ind in 1, size(*$cl[1]$*)]* $\times$ *1, size(*$cl[2]$*)]* $\times \ldots \times$ *1, size(*$cl[n]$*)]*
do
    $s \leftarrow \Theta$;
    for i = *1 to n* **do**
        $s \leftarrow s \cap cl[i][ind[i]]$;
    if $s = \emptyset$ **then**
        $lconf \leftarrow 1$;
        el is an empty expert;
        for i = *1 to n* **do**
            $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
            $el(cl[i][ind[i]]) \leftarrow el(cl[i][ind[i]]) + ex[i](cl[i][ind[i]])$;
        $sum \leftarrow 0$;
        for c *in* el **do**
            $sum \leftarrow sum + g(el(c))$;
        for c *in* el **do**
            $ep(c) \leftarrow ep(c) + g(el(c)) * lconf/sum$;
    else
        $lconf \leftarrow 1$;
        for i = *1 to n* **do**
            $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
        $ep(s) \leftarrow ep(s) + lconf$;